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SCALE COVARIANT GRAVITATION. VI. STELLAR STRUCTURE AND EVOLUTION

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ABSTRACT

Using the scale covariant theory of gravitation, we derive results of interest to stellar structure and evolution.

We first derive the radiative transport equation which finds an immediate application in the study of stellar evolution with a variable gravitational "constant" G.

We show that the Sun's absolute luminosity scales as $L \sim GM/\kappa$, where κ is the stellar opacity. Taking into account that the Earth-Sun distance is also affected by a variable G, it is then shown that the Earth's effective temperature scales like $T_{\rm eff} \sim (G^2M/\kappa)^{1/4}$. Comparisons with previous estimates, based on simple Newtonian framework, are presented and discussed.

A third result that follows from this analysis is that a polytropic equation of state of the form $p \sim \rho^\gamma$ is incompatible with a variable G, the correct result being $p \sim \rho^\gamma(G/\beta^2)^{\gamma-1}$, where β is the gauge function relating atomic to gravitational times.

Finally, we derive a formula for the age of globular clusters as a function of G, using the previously derived expression for the absolute luminosity.

Numerical estimates are presented.

Subject headings: gravitation — stars: evolution — stars: interiors

I. INTRODUCTION

In a series of papers (Canuto *et al.* 1977; Canuto and Hsieh 1979; Hsieh and Canuto 1981; hereafter Papers I, II, V), the scale covariant theory of gravitation, SCT, was developed so as to have a consistent framework to systematically study the astrophysical consequences of a varying gravitational constant G. A concise summary of the scale covariant theory can be found in Canuto, Hsieh, and Owen (1979*a*).

Cosmological data were analyzed in (Canuto, Hsieh, and Owen 1979*b*; Canuto and Owen 1979, hereafter Papers III, IV). With an imposed variation of the type $G \sim t^{-1}$ the SCT was found to be fully compatible with a wide range of observations (see also Canuto and Hsieh 1980*a*).

There have been many attempts in the past to use astronomical data such as the evolution of the solar luminosity to limit the possible variation of G. However, such previous analyses lacked a consistent theoretical framework. Their results can therefore give at best a qualitative indication of what phenomena one may expect but cannot be used to put restriction of the variation of G.

In this paper we study stellar structure and evolution in the framework of the SCT. A radiative transfer equation is derived from the kinetic theory presented in Paper V. This equation, combined with the hydrostatic equilibrium equation, yields a relation between the stellar luminosity and β, the scaling function directly related to a variable G (see eq. [10.3] below).

Application of the above relations allows us to study the evolution of the solar luminosity, its effects on the history of the surface temperature of the Earth, and finally the age of globular clusters. The "polytropic equations of state" are found to require modifications in the SCT. The effect on the Chandrasekhar mass and on the luminosity of white dwarfs is discussed.

II. FREE STREAMING RADIATION

It is well known that the transport equation is derivable as the first moment of the kinetic equations which reduces to the Liouville equation when sources and sinks are neglected. Thus, the results of Paper V, dealing with the kinetic theory in the context of the scale covariant gravitation, can be directly applied to give the free streaming part of the desired transport equation. For the purpose of displaying possible physical effects of a varying G on transport phenomena (through the gauge function β), it is useful to derive the equation in terms of an orthonormal basis so that the vector and tensor components with respect to this basis can be directly identified with observed variables.

Since the physical effects of the gauge function β will be our main concern, we shall study the explicit form of the transport equation for a *gravitationally flat metic* (Paper II, eq. [2.19]):

$$g = \beta^{-2}\eta_{\mu\nu}dx^\mu \otimes dx^\nu , \tag{2.1}$$

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where the symbol $\otimes$ denotes tensor product. As in Paper V, we shall use the compact notation of differential geometry for convenience.

The form (2.1) suggests a natural set of orthonormal bases $\{e_\mu\}$ with

$$e_\mu = \beta \frac{\partial}{\partial x^\mu}, \tag{2.2}$$

and a dual basis $\{\omega^\mu\}$ given by

$$\omega^\mu = \beta^{-1} dx^\mu, \tag{2.3}$$

so that

$$\omega^\mu e_\nu \equiv \delta_\nu{}^\mu, \tag{2.4}$$

and

$$g = \eta_{\mu\nu} \omega^\mu \otimes \omega^\nu. \tag{2.5}$$

It is the Minkowskian character of the metric with respect to this basis that gives the latter its particular physical significance. We also note that the tangent vectors e_μ are differential operators acting on functions of space-time. This underlines a peculiar problem in the comparison of the SCT with the Newtonian and special relativistic theory. In the latter, the *observed* time and distance intervals can be identified with coordinate intervals (dt, dx^i) so that the differential operators $(\partial/\partial t, \nabla_i = \partial/\partial x^i)$ can be considered a coordinate basis. This is no longer true in the SCT, even when space-time is gravitationally flat. Differentiations with respect to physical time intervals and with respect to physical distances do not commute in general. In fact, it can be easily seen from equation (2.2) that

$$[e_\mu, e_\nu] = e_\mu(\ln \beta) e_\nu - e_\nu(\ln \beta) e_\mu. \tag{2.6}$$

If we define the connection coefficients $^*\Gamma_{\mu\nu}{}^\lambda$ with respect to the basis $\{e_\mu\}$ by the relation

$$\nabla_{e_\mu} e_\nu = {}^*\Gamma_{\mu\nu}{}^\lambda e_\lambda \tag{2.7}$$

and take into account the compatibility condition on a Weyl manifold (Paper V, eq. [2.1])

$$\nabla g - 2\aleph \otimes g = 0, \tag{2.8}$$

and the torsionless condition

$$\nabla_{e_\mu} e_\nu - \nabla_{e_\nu} e_\mu - [e_\mu, e_\nu] = 0, \tag{2.9}$$

it can be shown that for general basis the following relation holds:

$$\begin{aligned} 2^*\Gamma_{\mu\nu}{}^\lambda = {} & g^{\lambda\rho}(e_\mu g_{\rho\nu} + e_\nu g_{\mu\rho} - e_\rho g_{\mu\nu}) - 2\aleph_\mu g_\nu{}^\lambda - 2\aleph_\nu g_\mu{}^\lambda + 2\aleph^\lambda g_{\mu\nu} \\ & + \omega^\lambda [e_\mu, e_\nu] - g^{\lambda\rho} g_{\mu\sigma} \omega^\sigma [e_\nu, e_\rho] - g^{\lambda\rho} g_{\nu\sigma} \omega^\sigma [e_\mu, e_\rho]. \end{aligned} \tag{2.10}$$

In the above equation, $g_{\mu\nu}$ are components of the metric with respect to the basis $\{\omega^\mu\}$, dual to $\{e_\mu\}$. Restricting our considerations to the system (2.2)–(2.5), we have

$$^*\Gamma_{\mu\nu}{}^\lambda = \delta_\nu{}^\lambda e_\mu \ln \beta. \tag{2.11}$$

The Liouville equation derived in Paper V (eq. [5.3]),

$$p^\mu \left[e_\mu - ({}^*\Gamma_{\mu\nu}{}^\lambda + \aleph_\mu g_\nu{}^\lambda) p^\nu \frac{\partial}{\partial p^\lambda} \right] f + (2 + \pi_g) \aleph_\nu p^\nu f = 0, \tag{2.12}$$

can be transformed to read

$$p^\mu e_\mu f - (2 + \pi_g) e_\mu \ln \beta p^\mu f = 0, \tag{2.13}$$

using the orthonormal basis defined by equations (2.2)–(2.5), along with equation (2.11) and the relation

$$\aleph_\mu = -e_\mu \ln \beta. \tag{2.14}$$

Equation (2.14) states that space-time is an integrable Weyl manifold and it is derivable from Paper I, equation (2.23). Taking the first moment of equation (2.13), we find

$$e_\nu T^{\mu\nu} - (2 + \pi_g) T^{\mu\nu} e_\nu \ln \beta = 0, \tag{2.15}$$

where $T^{\mu\nu}$ is the energy-momentum tensor.

To rewrite equation (2.15) in a more familiar form, we note that with respect to the basis $\{e_\mu\}$ the components of the energy momentum tensor can be written as

$$T^{\mu\nu} = \begin{pmatrix} \rho_\gamma\,, & \Phi^i \\ \Phi^j\,, & P^{ij} \end{pmatrix}\,, \tag{2.16}$$

where ρ_γ is the observed energy density, Φ^i is the observed momentum flux, and P^{ij} is the pressure tensor. Thus, identifying e_μ with $(\partial/\partial t, \nabla_j)$, equation (2.15) becomes

$$\frac{\partial}{\partial t}\rho_\gamma + \nabla_j \Phi^j - (2 + \pi_g)\left(\rho_\gamma \frac{\partial}{\partial t} + \Phi^j \nabla_j\right) \ln \beta = 0\,, \tag{2.17a}$$

$$\frac{\partial}{\partial t}\Phi^i + \nabla_j P^{ij} - (2 + \pi_g)\left(\Phi^i \frac{\partial}{\partial t} + P^{ij} \nabla_j\right) \ln \beta = 0\,. \tag{2.17b}$$

When sources and sinks are neglected, the above equations hold for arbitrary $T^{\mu\nu}$. In this paper, we shall only be interested in radiative transport. Hence, ρ_γ and Φ_i will be referred to as the radiative energy density and flux, respectively. The radiative pressure P^{ij} will be replaced by

$$P^{ij} = \tfrac{1}{3}\rho_\gamma \delta^{ij}\,, \tag{2.18}$$

known as the Eddington approximation.

When the gauge function β is constant, equation (2.17) is seen to represent simply the conservation of energy and momentum in special relativity. It can be demonstrated that with the β terms, equation (2.17) express the co-covariant conservation of the energy-momentum tensor for the given metric (2.1). This particular form of the equation is convenient for the discussion of the transport processes, since there are terms which correspond exactly to the standard streaming of the radiation. All modifications of the streaming due to SCT are collected in the β terms.

III. ABSORPTION AND EMISSION OF RADIATION

The right-hand side of the transport equation describing the emission and absorption of photons deals with the interaction between radiation and matter, a subject which has not yet been given a full description within the scale covariant framework. Consequently, the right-hand side of the equation cannot be derived from fundamental principles as we have done for the left-hand side in the previous section. For the purposes of this paper, we shall introduce the emission function and the absorption coefficient as phenomenological parameters, describing the properties of the medium through which radiation travels. Although these parameters can be introduced at the level of the kinetic equation, we shall do so directly at the level of the transport equation for simplicity. The extra details needed in the kinetic equation do not add to our understanding of these parameters, in the absence of a fundamental theory.

Thus, we add to the right-hand side of the transport equation (2.15) the effects of sources and sinks of radiation:

$$e_\mu T^{\mu\nu} - (2 + \pi_g) T^{\mu\nu} e_\nu \ln \beta = S^\mu - \kappa \rho T^{\mu\nu} u_\nu\,, \tag{3.1}$$

where S^μ and κ are the emission functions and the absorption coefficient, respectively. The symbol u_ν represents the components, with respect to the basis $\{e_\nu\}$, of the velocity u of the medium. If the medium is at rest, we can write

$$u = e_0 \to u_\nu = (1, 0, 0, 0)\,, \tag{3.2}$$

as can be seen from equation (3.18) of Paper II. We note that the factor of β in (Paper II, eq. [3.18]) has been absorbed by the orthonormal basis here. The source functions can also be simplified if we assume the emission to be isotropic in the rest frame of the medium. Then, in analogy with equation (3.2), we can stipulate

$$S^0 = S\,, \qquad S^i = 0\,, \tag{3.3}$$

so that equation (3.1) can be written explicitly as

$$\frac{\partial}{\partial t}\rho_\gamma + \nabla_j \Phi^j - (2 + \pi_g)\left(\rho_\gamma \frac{\partial}{\partial t} + \Phi^j \nabla_j\right) \ln \beta = S - \kappa \rho \rho_\gamma\,, \tag{3.4a}$$

$$\frac{\partial}{\partial t}\Phi^i + \nabla_j P^{ij} - (2 + \pi_g)\left(\Phi^i \frac{\partial}{\partial t} + P^{ij} \nabla_j\right) \ln \beta = -\kappa \rho \Phi^i\,. \tag{3.4b}$$

The functions S and κ are determined by the physical condition of the medium. If the latter is in local thermodynamic equilibrium, they are, in general, a function of its density and temperature. However, within the SCT, we have no reason to rule out a possible dependence of S and κ on the gauge function β.

IV. RADIATIVE TRANSFER EQUATION

When we consider a stationary, spherically symmetric outflow of radiation, the relevant transport equation can be obtained from equation (3.4b):

$$\frac{1}{3}\frac{\partial}{\partial r}\rho_\gamma - (2 + \pi_g)\Phi \frac{\partial}{\partial t} \ln \beta = -\kappa\rho\Phi , \tag{4.1}$$

where Φ is the radial (the only nonvanishing) component of the radiative flux. In equation (4.1), it has been assumed that all spatial derivatives of β vanish and equation (2.18) holds. Furthermore, t and r are the physical time and radial variables.

Replacing the flux per square centimeter Φ with the luminosity function at radius r,

$$L_r = 4\pi r^2 \Phi , \tag{4.2}$$

equation (4.1) takes the form

$$L_r = -\frac{4\pi r^2}{3\kappa\rho}\frac{1}{1-(1/\kappa\rho)(2+\pi_g)(\dot{\beta}/\beta)}\frac{d\rho_\gamma}{dr} . \tag{4.3}$$

Since the quantity $1/\kappa\rho$, representing the diffusion time across the star, is several orders of magnitude smaller than the age of the universe ($\beta/\dot{\beta} \sim t$), we shall write

$$L_r = -\frac{4\pi r^2 a}{3\kappa\rho}\frac{\beta^2}{G}\frac{d}{dr} T^4 , \tag{4.4}$$

where a is the blackbody constant and we have used the relation (Paper II, eq. [5.56])

$$\rho_\gamma = a \frac{\beta^2}{G} T^4 . \tag{4.5}$$

The new radiative transfer equation (4.4) together with the hydrostatic equilibrium equation

$$\frac{dp}{dr} = -G\frac{m(r)\rho(r)}{r^2} , \tag{4.6}$$

which has already been shown to hold unaltered (Paper I, eq. [4.42]) in the SCT, constitute the two basic relations needed to study the effects of $G = G(t)$ on stellar evolution.

V. STELLAR LUMINOSITY

a) Previous Work

Historically, the first test ever proposed to check the astrophysical consequences of a varying G had to do with stellar evolution. In particular, Teller (1948) studied the evolution of the Sun's luminosity. His work was followed by detailed numerical investigations (Pochoda and Schwarzschild 1964; Roeder and Demarque 1966; Roeder 1967; vandenBergh 1976, 1977; Maeder 1977*a*, *b*; Carigan, Beaudet, and Sirois 1979) which confirmed Teller's original analytic treatment and extended it to include the possibility of matter creation.

Since *all* these analyses were performed before the formulation of the SCT, they did not contain the differentiation between gravitational and atomic time intervals Δt_E and Δt,

$$\Delta t_E = \beta(t)\Delta t , \tag{5.1}$$

which is central to the present theory. One can therefore characterize these early investigations as having β = constant. Furthermore, it was assumed that the variation of G can be fully accounted for by substituting $G \to G(t)$, where G *already* appears. Since in the standard stellar evolution equations, G appears only in equation (4.6), it was assumed that *all* the remaining equations were unaffected by the transition to a varying G. Such an assumption is, however, invalid, since even with β = constant, the radiative transfer equation is modified by the presence of G, (eq. [4.4), a factor not included in any of the previous analyses. Furthermore, account was not taken of a constraint valid in both the Newtonian framework, β = constant, as well as in the SCT. The constraint, which has been derived in different ways (Paper I, eq. [2.49]; Canuto, Hsieh, and Owen 1979*a*) is

$$\beta G M = G_0 M_0 = \text{constant} , \tag{5.2}$$

where the subscript zero stands for today, $\beta_0 = 1$, and β, G, and M are arbitrary functions of time. It can be seen that even with β = constant, equation (5.2) imposes a restriction between G and M, which reduces Teller's result for the stellar luminosity (Canuto and Hsieh 1980*b*),

$$L \sim \frac{1}{\kappa} G^4 M^3 , \tag{5.3}$$

to

$$L \sim \frac{1}{\kappa} G \,. \tag{5.4}$$

Furthermore, it is easy to see that if account is taken of the factor of G in equation (4.5), the remaining G in equation (5.4) cancels out, so that the final result reads

$$L \sim \frac{1}{\kappa} \,, \tag{5.5}$$

instead of equation (5.3) (Canuto and Hsieh 1980*b*). The fact that the G dependence disappears from equation (5.3) indicates that in the standard Newtonian framework G does not have a separate identity from the mass; it appears multiplied by M (or by ρ_γ, eq. [4.5]), *and the product GM is dynamically constrained to remain constant.* A renormalization of M can be made which makes G disappear from the problem.

b) The Scale Covariant Framework

We shall now derive the dependence of the absolute luminosity L on β and G in the scale covariant theory. Using equation (4.6) and the equation of state (Paper II, eq. [4.25]),

$$p = nkT \,, \tag{5.6}$$

we derive

$$\mathscr{R}T \sim GM \sim \frac{1}{\beta} \,, \tag{5.7}$$

where $\mathscr{R}$ is the radius of the star. We note that equation (5.7) holds true also for the equation of state $p_\gamma = \frac{1}{3}\rho_\gamma$, with ρ_γ given by equation (4.5). From equations (4.4) and (5.7), we now deduce

$$L \sim \frac{1}{\kappa M} (\mathscr{R}T)^4 \frac{\beta^2}{G} \sim \frac{(GM)^4}{\kappa M} \frac{\beta^2}{G} \sim (GM)^3 \frac{\beta^2}{\kappa} \,. \tag{5.8}$$

Using equation (5.2) we finally derive

$$L \sim \frac{1}{\kappa \beta} \,. \tag{5.9}$$

For constant β, we recover the Newtonian limit (5.5), as expected.

Since the opacity coefficient may depend on β, we shall write

$$\kappa = \kappa_{\rm st} \, f(\beta) \,, \tag{5.10}$$

where $\kappa_{\rm st}$ is the *standard* opacity and the function $f(\beta)$ represents the effects of β (and therefore of $G \sim \beta^{-g}$). Equation (5.9) now becomes

$$L = \frac{L_{\rm st}}{\beta f(\beta)} \,, \tag{5.11}$$

where $L_{\rm st}$ is the luminosity as from standard theory. Because of the strong dependence on the mean molecular weight μ, $L_{\rm st} \sim \mu^8$, $L_{\rm st}$ is not constant in time (Table 1).

From the luminosity relation equation (5.11), we can derive an expression for the effective surface temperature of the Sun, T_*, and for the average photon energy $\langle \epsilon_\gamma \rangle$ corresponding to thermal radiation of temperature T_*. T_* is defined by ($G \equiv G/G_0$)

$$L = 4\pi\mathscr{R}^2 \sigma \frac{\beta^2}{G} T_*^{\,4} \,, \tag{5.12}$$

where σ is the Stefan-Boltzmann constant. Hence,

$$T_* = T_0 \left[\frac{L_{\rm st}}{L_0} \frac{G}{\beta^3 f} \left(\frac{\mathscr{R}_0}{\mathscr{R}} \right)^2 \right]^{1/4} \,, \tag{5.13}$$

where T_0 is the present effective temperature of the Sun. Since the solar radius $\mathscr{R}$ is not constant even in standard theory, we shall write

$$\mathscr{R} = \mathscr{R}_{\rm st} \, r(\beta) \,, \tag{5.14}$$

TABLE 1

ABSOLUTE LUMINOSITY AND EFFECTIVE TEMPERATURE OF THE SUN (as from standard theory)[a]

t[b]	L_{st}/L_0	T_{*st} (K)
1.20	0.908	5630
1.83	0.866	5600
2.48	0.827	5580
3.14	0.793	5550
3.79	0.764	5540
4.12	0.747	5530
4.28	0.736	5528
4.36	0.721	5506
4.40	0.709	5460

[a] R. Stothers, private communication.
[b] 10^9 yr ago.

where $\not{r}(\beta)$ represents the effects of β (and G) on the equilibrium stellar radius. The final result is

$$T_* = T_{*st}\left(\frac{G}{\beta^3}\frac{1}{\not{r}^2}\right)^{1/4}, \tag{5.15}$$

with

$$T_{*st} = T_0\left[\frac{L_{st}}{L_0}\left(\frac{\mathscr{R}_0}{\mathscr{R}_{st}}\right)^2\right]^{1/4}.$$

In equation (5.15) we have now isolated the contribution from the SCT as deriving from thermodynamics, opacity and dynamics. Typical values of L_{st} and T_{*st} are presented in Table 1.

The average photon energy is given directly by the temperature:

$$\langle \epsilon_\gamma \rangle = kT_* , \tag{5.16}$$

where k is the Boltzmann constant. We caution that translation of the above equation to one relating the average wavelength and the radiation temperature should be done with great care, as the photon energy-frequency relation has been modified in the SCT (Paper II, eq. [3.15]). However, since the photon energy is physically more meaningful, we shall limit our considerations to $\langle \epsilon_\gamma \rangle$.

VI. THE EFFECTIVE TEMPERATURE OF THE EARTH

With the solar luminosity given by equation (5.11), we can calculate the effective surface temperature of the Earth. Let D be the Earth-Sun distance. We deduce

$$T_{eff} = (T_{eff})_{st}\left[\frac{G}{\not{r}\beta^3}\left(\frac{D_0}{D}\right)^2\right]^{1/4}. \tag{6.1}$$

Since the distance D is determined solely by gravity, we have the exact result (Canuto, Hsieh, and Owen 1979*a*; see also eq. [5.1]):

$$D_E = \beta D = \text{constant} , \tag{6.2}$$

so that

$$T_{eff} = (T_{eff})_{st}\left(\frac{G}{\not{r}\beta}\right)^{1/4}, \tag{6.3}$$

with

$$(T_{eff})_{st} = T_{eff}{}^0\left(\frac{L_{st}}{L_0}\right)^{1/4}. \tag{6.4}$$

The relevance of these results for the problem of the origin of life will be discussed in § XI.

VII. THE FUNCTIONS $\ell(\beta)$ AND $\imath(\beta)$

In § V, we introduced the functions ℓ and $\imath$ of β representing the effects of G-variation on stellar opacity and radius. In this section, we shall sketch their physical significance. First, we consider the function $\imath$. It will be shown in § VIII that the first law of thermodynamics consistent with the SCT implies the following relation between pressure p and density ρ:

$$p \propto \rho^{\gamma}\left(\frac{G}{\beta^2}\right)^{\gamma-1}, \tag{7.1}$$

where γ is a constant index, given by equation (8.17). Once equation (7.1) is substituted into the hydrostatic equilibrium equation (4.6), one obtains that the radius $\mathscr{R}$ scales as

$$\mathscr{R} \sim \frac{1}{\beta}; \tag{7.2}$$

i.e., from equation (5.14)

$$\imath(\beta) \equiv \beta^{-1}, \tag{7.3}$$

so that

$$T_* = T_{*\mathrm{st}}\left(\frac{G}{\ell\beta}\right)^{1/4}, \tag{7.4}$$

Note that equation (7.2) states that the solar radius scales exactly like a gravitational length, as dictated by equation (6.2). This may seem surprising at first. Since nongravitational forces are involved in determining the equilibrium radius of the Sun, one would expect $\mathscr{R}$ and D to scale differently. However, for our polytropic gas model, the matter pressure is derived from the kinetic motion of the constituent particles. Hence, we are not dealing with a problem having truly mixed dynamics since the kinetic motion is not governed by nongravitational dynamics. This need not always be the case as Canuto (1981) has shown in the study of the evolution of the radius of the Earth.

Next, we consider the function ℓ. The opacity coefficient κ may be defined as

$$\kappa\rho_{\gamma} = \varepsilon, \tag{7.5}$$

where the left-hand side represents the amount of energy absorbed (ρ_γ being the radiation energy density) and the right-hand side represents the energy radiated by, for example, thermal bremsstrahlung. We shall write the emissivity ϵ as

$$\epsilon \propto \rho T^{\xi}\ell_{\epsilon}(\beta), \tag{7.6}$$

where the subscript ϵ reminds us of the physical origin of the function ℓ_ϵ. We note that if $\ell_\epsilon = 1$, ϵ reduces to the familiar emissivity for bremsstrahlung computed from standard physics. The function ℓ_ϵ, representing the effects of β on atomic phenomena, cannot yet be computed from first principles at the present stage of development of the SCT.

Using equations (4.5), (5.2), and (5.7), we obtain from equations (7.5) and (7.6)

$$\kappa \propto T^{\xi-1}\ell_{\epsilon}(\beta), \tag{7.7}$$

which for the so-called modified Kramers opacity, $\xi = 1$, reduces to

$$\kappa = k_{\mathrm{st}}\,\ell_{\epsilon}(\beta). \tag{7.8}$$

In the process of deriving equation (7.8), there has been a very interesting cancellation due to equation (5.2) of the β^2/G factor arising from equation (4.5). We therefore conclude that the function $\ell(\beta)$ is just $\ell_\epsilon(\beta)$ arising from the β dependence of the atomic interactions needed to evaluate the emissivity ϵ. In what follows, ℓ_ϵ will be denoted simply by ℓ.

VIII. POLYTROPES AND THE CHANDRASEKHAR MASS

In the theory of late stellar structure, a major role has been played by the study of polytropes, i.e., stars for which the relation between pressure and density is of the form

$$p = K\rho^{\gamma}, \tag{8.1}$$

where γ and K are constants. With an equation of the state of the form (8.1), the hydrostatic equilibrium equation (4.6) can be solved exactly through the well-known Lane-Emden functions. The most interesting relation is the one relating the mass M, the radius $\mathscr{R}$, and the gravitational constant G. The well-known result is (Chandrasekhar 1937)

$$M \propto \mathscr{R}^{(3\gamma-4)/(\gamma-2)}\left(\frac{K}{G}\right)^{-1/(\gamma-2)}. \tag{8.2}$$

For a polytrope with $\gamma = \frac{4}{3}$, equation (8.2) leads to a result independent of the radius, namely

$$M \propto \left(\frac{K}{G}\right)^{3/2} . \tag{8.3}$$

In standard theory, where K is expressed in terms of the Planck constant $\hbar$ and an atomic mass m only, one derives the so-called Chandrasekhar mass,

$$M_C \propto \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m^2} . \tag{8.4}$$

If one assumes that equation (8.4) holds true in a G-varying scheme, one reaches the conclusion that

$$M_C \propto G^{-3/2} . \tag{8.5}$$

However, this is not true. In the following, we shall show that within the SCT the equation of state (8.1) is *not compatible with the first law of thermodynamics*, and it must be replaced by

$$p \propto \rho^{\gamma}\left(\frac{G}{\beta^2}\right)^{\gamma - 1} . \tag{8.6}$$

From (Paper II, eq. [4.8]), we have the modified first law of thermodynamics

$$\theta dS = dU + pdV + [(1 - g)U + 3pV]d\phi , \tag{8.7}$$

where (Paper II, eqs. [4.6; 4.31])

$$\theta \sim T\beta^{g-1} , \qquad G \sim \beta^{-g} , \qquad \phi = \ln \beta . \tag{8.8}$$

Using the equation of state (Paper II, eqs. [4.25, 4.27])

$$p = nkT = \frac{N}{V} kT = \frac{N_0}{V} \beta^{g-1} kT ,$$

$$pV = \beta^{g-1} RT , \tag{8.9}$$

where we have used the fact that the total number of particles N scales as (Paper II, eq. [4.26])

$$N = N_0 \beta^{g-1} , \tag{8.10}$$

and the expression for the total energy (Paper II, eq. [4.32]),

$$U = \frac{1}{\Gamma} NkT , \tag{8.11}$$

one can transform equation (8.7) to read

$$\frac{TdS}{dT} = \frac{R}{\Gamma} + R\left(\frac{dV}{V}\right)\left(\frac{dT}{T}\right)^{-1} + 3R\left(\frac{d\beta}{\beta}\right)\left(\frac{dT}{T}\right)^{-1} , \tag{8.12}$$

where R is the gas constant. As in the standard case, we denote

$$T\frac{dS}{dT} \equiv c , \tag{8.13}$$

with $c = 0$ for an *adiabatic* transformation and $c = \infty$ for an *isothermal* one. Using again equation (8.9) to transform away dT/T, one obtains

$$pV^{1 + 1/n} \propto \beta^{g - 1 - 3/n} , \tag{8.14}$$

where the polytropic index n is defined as

$$n \equiv \frac{1}{\Gamma} - \frac{c}{R} . \tag{8.15}$$

The other polytropic relations are

$$TV^{1/n} \sim \beta^{-3/n} , \qquad pT^{-(1+n)} \sim \beta^{2+g} . \tag{8.16}$$

Eliminating the volume V in favor of the rest-mass density ρ, we obtain from equation (8.14)

$$p \propto \rho^{1 + 1/n}\left(\frac{G}{\beta^2}\right)^{1/n} ,$$

which with the substitution,

$$\gamma = 1 + \frac{1}{n} . \tag{8.17}$$

can finally be rewritten as

$$p \propto \rho^{\gamma}\left(\frac{G}{\beta^2}\right)^{\gamma - 1} . \tag{8.18}$$

Note that the new factor in equation (8.18) disappears in the isothermal case, $\gamma = 1$. In the adiabatic case

$$c = 0 , \quad n = \frac{1}{\Gamma} , \quad \gamma = 1 + \Gamma , \tag{8.19}$$

we have already shown that the scaling law for an ideal classical gas is (Paper II, eq. [4.35])

$$TV^{\Gamma}\beta^{3\Gamma} = \text{constant} . \tag{8.20}$$

Using equation (8.20) in equation (8.9), we can indeed check that equation (8.18) holds with $\gamma = 1 + \Gamma$.

Note also that from the scaling properties of energy density and pressure (Paper I, eq. [2.11c])

$$\begin{aligned}(p, \rho) &\propto (p_E, \rho_E)\beta^{2+g} \\ &\propto (p_E, \rho_E)\left(\frac{\beta^2}{G}\right) ,\end{aligned} \tag{8.21}$$

we obtain from equation (8.18)

$$p_E \propto \rho_E{}^{\gamma} , \tag{8.22}$$

as should be expected because in gravitational units the first law of thermodynamics (eq. [8.7]) reduces to the standard one, and equation (8.22) must follow. This result indicates once more that *the assumption that gravitational physics is unchanged naturally leads to changes at the atomic level.* Equations (4.1) and (4.5) are other examples of such modifications.

With equation (8.18) replacing equation (8.1), the new relation between M, $\mathscr{R}$, and G reads

$$M \sim \mathscr{R}^{(3\gamma-4)/\gamma-2}G^{1/\gamma(-2)}\left(\frac{G}{\beta^2}\right)^{-(\gamma-1)/(\gamma-2)} . \tag{8.23}$$

For $\gamma = \frac{4}{3}$, we have instead of equation (8.4)

$$M_{\mathrm{C}} \sim \left(\frac{hc}{G}\right)^{3/2}\frac{1}{m^2}\left(\frac{G}{\beta^2}\right)^{1/2} \tag{8.24}$$

as the new expression for the Chandrasekhar mass. Equation (8.24) can alternatively be written

$$\beta G M_{\mathrm{C}} \sim \frac{(hc)^{3/2}}{m^2} . \tag{8.25}$$

In atomic units, the right-hand side is constant by definition, and so is the left hand side because of equation (5.2). The result is therefore consistent. *The Chandrasekhar mass does not scale like* $G^{-3/2}$, *but like* $(\beta G)^{-1}$, *as any other mass in this theory.*

IX. LUMINOSITY OF WHITE DWARFS

The possible changes brought about by a varying G on the luminosity of degenerate white dwarfs has been a subject of a recent study (Mansfield and Malin 1980). The authors, employing an equation of state of the form

$$p \propto \rho^{\gamma} , \tag{9.1}$$

and their definition of the luminosity, i.e.,

$$L = -E\frac{\partial}{\partial t}\ln(\beta G E) , \quad E \sim \frac{GM^2}{\mathscr{R}} , \tag{9.2}$$

concluded that such serious problems would ensue for the βG = constant gauge, as to make it unacceptable. Such a conclusion is, however, dependent on the use of equation (9.1) which, as we have shown, is not consistent with the first law of thermodynamics in the SCT. If we use expression (8.6), i.e., equation (8.23), we obtain

$$E \sim \frac{GM^2}{\mathscr{R}} \sim \frac{1}{\beta G}E_{\mathrm{st}} , \tag{9.3}$$

where E_{st} is the energy computed from standard theory. It then follows from equation (9.2) that

$$L = \frac{1}{\beta G} L_{st} . \tag{9.4}$$

Since the luminosity L that we compare with observations is evaluated today, when $\beta_0 G_0 = 1$, we have

$$L_0 = L_{st} \tag{9.5}$$

for *any gauge*.

Contrary to the conclusion of Mansfield and Malin, today's luminosity of degenerate stars is identical to the one derived from standard theory for any combination of β and G, and it cannot therefore be used as a discriminant for or against a time variation of G as long as one adopts the first of equation (9.2).

X. THE AGES OF STARS AND CLUSTERS

One important consequence of a larger G in the past is that the true age of stars $\Delta t(= t_0 - t)$ can be considerably different from the standard age Δt_{st} computed with G = constant.

We have from equations (5.11) and (5.2)

$$\Delta t_{st} = \int_{t_0 - \Delta t}^{t_0} dt \frac{G(t)}{f(\beta)} , \tag{10.1}$$

where t_0 is the age of the Universe in atomic units. Since we do not yet possess an explicit function $f(\beta)$, we shall use a simple parametrization of the form

$$f(\beta) = \beta^s(t) . \tag{10.2}$$

At the same time, we shall use the power laws

$$G \sim \beta^{-\pi_g} , \qquad \beta(t) = \left(\frac{t}{t_0}\right)^{-\epsilon} , \ \pi_g \equiv g , \tag{10.3}$$

employed in all our previous computations (Papers I–IV). The different gauges used were

$$\epsilon = -1 , \qquad -\tfrac{1}{2} , \qquad +\tfrac{1}{2} , \qquad +1 . \tag{10.4}$$

Since a positive ϵ seems to be ruled out by recent data on the Moon (Canuto and Hsieh 1980*a*), we shall only consider $\epsilon < 0$. The simplest case, $\epsilon = -1$, corresponds to

$$G\beta = \text{constant} , \qquad G \sim t^{-1} , \tag{10.5}$$

and is called the no-matter creation case (Dirac 1937). The case $\epsilon = -\frac{1}{2}$, corresponding to

$$G\beta^2 = \text{constant} , \tag{10.6}$$

was proposed by the present authors (Canuto and Hsieh 1978) in connection with the study of the 3 K blackbody radiation.

Integrating equation (10.1), we obtain the relation between the G-dependent age Δt and the standard one Δt_{st}

$$\frac{\Delta t_{st}}{t_0} = \frac{1}{1 + \epsilon(g + s)} \left[1 - \left(1 - \frac{\Delta t}{t_0}\right)^{1 + \epsilon(g + s)}\right] . \tag{10.7}$$

XI. A PARTICULAR CASE

The general expressions we derived above for the stellar luminosity, the effective surface temperature of the Sun, the wavelength at maximum radiation intensity, the effective surface temperature of the Earth, and the age of globular clusters all depend on the opacity coefficient κ, which in turn may be affected by a variable G through the function $f(\beta)$. As we have explained, the function $f(\beta)$ represents modifications on the atomic dynamics demanded by a G-varying framework. Since we have not yet completed the atomic part of the SCT, we shall assume here $f(\beta) = 1$, so that we can isolate and study the consequences of the β dependence originating from nonatomic contributions.

We shall employ the simplest of all gauges, namely $\epsilon = -1$; i.e.,

$$\beta G = \text{constant} , \qquad M = \text{constant} . \tag{11.1}$$

TABLE 2

AGES (in 10^9 yr) OF GLOBULAR CLUSTERS WITH AND WITHOUT A VARIABLE G
($\bar{q}_0 = 0$, $H_0 = 80$ km s^{-1} Mpc^{-1})

Object	Δt_{st}[a]	Δt[b]
47 Tuc	11.5[c]	7.45
M15	15.0[c]	8.64
M67	9[d]	6.37

[a] G = constant.
[b] $G \neq$ constant.
[c] Demarque and McClure (1977).
[d] Dicke (1962).

We derive from equations (5.11), (7.4), (6.3), and (10.7)

$$L \sim \frac{1}{\beta \not{f}} \sim G \,, \tag{11.2}$$

$$T_* \sim \left(\frac{G}{\not{f}\beta}\right)^{1/4} \sim G^{1/2} \,, \tag{11.3}$$

$$\langle \epsilon_\gamma \rangle \sim G^{1/2} \,, \tag{11.4}$$

$$T_{\rm eff} \sim \left(\frac{G}{\not{f}\beta}\right)^{1/4} \sim G^{1/2} \,, \tag{11.5}$$

$$\Delta t_{st} = \frac{t_0}{1 + \epsilon g}\left[1 - \left(1 - \frac{\Delta t}{t_0}\right)^{1+\epsilon g}\right] . \tag{11.6}$$

In particular, using the form $G = G_0(t_0/t)$ employed in all the computations so far we conclude that:

1) 4.5×10^9 yr ago, the Sun's absolute luminosity was 43%–33% larger than the standard value L_{st}, depending on the age of the Universe, $t_0 = 15 \times 10^9$ yr or $t_0 = 18 \times 10^9$ yr.

2) The Sun's effective temperature was also 19%–15% larger than the standard value, again depending on the age of the Universe.

3) The average photon energy emitted by the Sun was 19%–15% greater than in standard theory.

4) Standard theory predicts a lowering of ~ 8% in the Earth's effective temperature 4.4×10^9 yr ago, as from equation (6.4) and Table 1. The falling of the temperature below the freezing point of sea water between $(2\text{–}3) \times 10^9$ yr ago creates difficulties because of the evidence of life and therefore of liquid water all the way to 3.5×10^9 yr ago. While it is possible to alter the components of the Earth's atmosphere so as to enhance the greenhouse effects and keep the Earth from freezing (Sagan and Muller 1972; Sagan 1977; Owen, Cess, and Ramanathan 1979), the present theory offers a different solution. From equation (11.5) it follows that 4.4×10^9 yr ago, $T_{\rm eff}$ was 19%–15% larger than $(T_{\rm eff})_{st} \sim 0.92 T_{\rm eff}{}^0$, so that $T_{\rm eff} \sim (1.09\text{–}1.06)\, T_{\rm eff}{}^0$. The present theory therefore yields an Earth's past effective temperature 6%–9% higher than today, thus avoiding the problem mentioned above without having to resort to changes in the chemistry of the early atmosphere.

5) As for the globular clusters ages, the choice $\epsilon g = -1$ reduces equation (11.6) to

$$\Delta t = t_0\left[1 - \exp\left(-\frac{\Delta t_{st}}{t_0}\right)\right] , \tag{11.7}$$

an expression evaluated in Table 2 for three well-known cases.

Table 2 clearly indicates that the cluster ages are considerably reduced by a variable G. Since it has recently been advocated that the Hubble's constant should be rescaled upward, there may be an age discrepancy especially in the case of M15. If $H_0 = 80$ km s^{-1} Mpc^{-1}, the age of the Universe is *at most* 12.5×10^9 yr, too short compared to 15×10^9 yr computed from M15 (Table 2). While a solution has been proposed which implies a nonzero cosmological constant (Tinsley 1978), a larger G in the past reduces such ages well below the age of the Universe, thus eliminating the possible discrepancy.

XII. SUMMARY OF THE RESULTS

In this paper we have derived the radiative transfer equation in the scale covariant theory of gravitation, SCT. The exact result is contained in equation (2.17a, b).

Once the absorption and emission coefficients are included, the radiative transfer equation of interest in astrophysics is given by equation (4.1), which we then write in the more familiar form (4.4).

The first comment about this equation concerns the presence of G. Even within a primitive theory with only G varying but otherwise no difference between gravitational and atomic times, i.e., with constant β, the radiative transfer equations has to be modified. Such modification was not taken into account in the pre–SCT investigations of the effects of G on stellar evolution, a topic that attracted a great deal of attention as a possible problem for the varying G hypothesis (Canuto and Hsieh 1980*b*).

We have shown that even in the pre–SCT framework, the correct result for the dependence of the absolute luminosity L is

$$L \sim \frac{1}{\kappa} \tag{12.1}$$

and not

$$L \sim \frac{G^4 M^3}{\kappa}\,, \tag{12.2}$$

as often stated. The disappearance of the factor G^4M^3 is due to two reasons: (*a*) the factor of G appearing in equation (4.4), and (*b*) the constraint $GM =$ constant, equation (5.2), which even with constant β still imposes a restriction between G and M. The two Dirac's hypothesis, $G \sim t^{-1}$ and $M =$ constant or $G \sim t^{-1}$ and $M \sim t^2$, cannot therefore be incorporated in the standard framework with constant β since they violate the constraint $GM =$ constant. To study such possibilities, one needs a new theoretical scheme, and this is the reason why we constructed the SCT, a framework which allows atomic and gravitational clocks to evolve differently during the expansion of the Universe; $\beta(t)$ is the new function that appears in the SCT.

Within this new framework, the absolute luminosity of a star is given by equation (5.9). This exact result allows us to evaluate the Sun's effective temperature, equation (5.15), and the corresponding average photon energy, equation (5.16). Another quantity of interest is the Earth's effective temperature, equation (6.3).

We have also presented expressions giving the age of stars and/or globular clusters taking into account the larger value of G in the past, equation (10.7).

Finally, and quite independently of gauge conditions, we have studied the behavior of polytropes with particular emphasis to the G dependence of the Chandrasekhar mass. It was shown that due to the change in the energy conservation law, the pressure versus density relation is altered: the equation of state is no longer $p \sim \rho^{\gamma}$, but $p \sim \rho^{\gamma}(G/\beta^2)^{\gamma-1}$. Therefore, the Chandrasekhar mass M_C does not scale as $G^{-3/2}$, but as $M_C \sim (\beta G)^{-1}$. Since we have not yet evaluated the possible β dependence of the opacity coefficient, *we cannot present numerical estimates for the quantities of interest.* The examples discussed serve to quantify the nonatomic β-dependence of the quantities in question.

REFERENCES

Canuto, V. M. 1981, *Nature*, **290**, 739.
Canuto, V. M., Adams, P. J., Hsieh, S.-H., and Tsiang, E. 1977, *Phys. Rev., D.*, **16**, 1643 (Paper I).
Canuto, V. M., and Hsieh, S.H. 1978, *Ap. J.*, **224**, 302.
———. 1979, *Ap. J. Suppl.*, **41**, 243 (Paper II).
———. 1980*a*, *Phys. Rev. Letters*, **44**, 695.
———. 1980*b*, *Ap. J.*, **237**, 613.
Canuto, V. M., Hsieh, S.-H., and Owen, J. R. 1979*a*, *M.N.R.A.S.*, **188**, 829.
———. 1979*b*, *Ap. J. Suppl.*, **41**, 263 (Paper III).
Canuto, V. M., and Owen, J. R. 1979, *Ap. J. Suppl.*, **243**, 301 (Paper IV).
Carigan, C., Beaudet, G., and Sirois, A. 1979, *Astr. Ap.*, **75**, 291.
Chandrasekhar, S. 1937, *An Introduction to the Study of Stellar Structure* (New York: Dover).
Demarque, P., and McClure, R. D. 1977, in *Evolution of Galaxies and Stellar Populations* ed. B. M. Tinsley, and R. B. Larsen (New Haven: Yale University Observatory), pp. 199–218.
Dicke, R. H. 1962, *Rev. Mod. Phys.*, **34**, 110.
Dirac, P. A. M. 1937, *Nature*, **139**, 323.
Hsieh, S.-H., and Canuto, V. M. 1981, *Ap. J.*, **248**, (Paper V).
Maeder, A. 1977*a*, *Astr. Ap.*, **56**, 359.
———. 1977*b*, *Astr. Ap.*, **57**, 125.
Mansfield, V., and Malin, S. 1980, *Ap. J.*, **237**, 349.
Owen, T., Cess, R. D., and Ramanathan, V. 1979, *Nature*, **227**, 640.
Pochoda, P., and Schwarzschild, M. 1964, *Ap. J.*, **189**, 587.
Roeder, R. C. 1967, *Ap. J.*, **149**, 131.
Roeder, R. C., and Demarque, P. 1966, *Ap. J.*, **144**, 1016.
Sagan, C. 1977, *Nature*, **269**, 224.
Sagan, C., and Muller, G. 1972, *Science*, **177**, 52.
Teller, E. 1948, *Phys. Rev.*, **73**, 801.
Tinsley, B. M. 1978, *Nature*, **273**, 208.
vandenBergh, D. A. 1976, *M.N.R.A.S.*, **176**, 455.
———. 1977, *M.N.R.A.S.*, **181**, 695.

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